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ON THE CONSTRUCTION OF SOLUTIONS OF QUASILINEAR NONAUTONOMOUS SYSTEMS IN RESONANCE CASES

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We consider a system with n degrees of freedom, of the following form:

$$x_s^{\cdot} = -\lambda_s y_s + \mu X_{s1}(x, y, t) + \mu^2 X_{s2}(x, y, t) + \ldots + f_{s0}(t) + \mu f_{s1}(t) + \ldots \tag{0.1}$$
$$y_s^{\cdot} = \lambda_s x_s + \mu Y_{s1}(x, y, t) + \mu^2 Y_{s2}(x, y, t) + \ldots + \varphi_{s0}(t) + \mu\varphi_{s1}(t) + \ldots$$
$$x \equiv (x_1, \ldots, x_n), \quad y \equiv (y_1, \ldots, y_n) \qquad (s = 1, \ldots, n)$$

Here $X_{s1}, \ldots, Y_{s1}, \ldots$ are polynomials of an arbitrarily high degree in x and y with continuous coefficients which are 2π-periodic in t. The functions $f_{s0}, \ldots, \varphi_{s0}, \ldots$ are continuous and have the same period. Quantity μ is a small parameter. We assume that both internal and external resonance are present in the system.

There exist various well worked out methods of investigating the oscillations of quasilinear nonautonomous systems in resonance cases (method of small parameter, method of averaging, e. a.). these reduce the problem of constructing the oscillations accurate to the first degree of the small parameter to obtaining solutions of, so called, fundamental (generating) amplitude equations. In the case of a system with several degrees of freedom, these equations represent a system of nonlinear algebraic equations, for which general solution does not esist. Thus, one problem leads to another which is no less complex.

In the present paper we use the results of [1, 2] to develop a method of constructing both periodic and almost-periodic solutions. This allows us to obtain the values of the fundamental amplitudes from a system of linear algebraic equations, when the order of the highest form accompanying μ is not greater than three. If X_{s1} and Y_{s1} contain terms of the order higher than three, then the equations defining the fundamental amplitudes will be also nonlinear, but simpler than those appearing in the method of small parameters, method of averaging, etc.

In contrast to [1], we do not assume the existence of a unique 2π-periodic solution for the system (0.1) with $f_{s1} \equiv \ldots \equiv \varphi_{s1} \equiv \ldots \equiv 0$ tending to the generating solution as $\mu \to 0$.

1. We shall construct, for (0.1), a solution [1] of the form

$$x_s = x_s^* + \xi_s^*, \qquad y_s = y_s^* + \eta_s^* \tag{1.1}$$

Here x_s^* and y_s^* denote those solutions of (0.1) which become the generating ones when $\mu = 0$ and which are usually obtained in the form of series

$$x_s^* = x_{s0}^*(t) + \mu x_{s1}^*(t) + \ldots, \qquad y_s^* = y_{s0}^*(t) + \mu y_{s1}^*(t) + \ldots$$

with periodic coefficients, while ξ_s^* and η_s^* are bounded solutions of the system

$$\begin{aligned} \xi_s^{*\cdot} &= -\lambda_s \eta_s^* + \mu\, \Xi_{s1}(\xi^*, \eta^*, M, t) + \mu^2 \Xi_{s2}(\xi^*, \eta^*, N, t) + \ldots \\ \eta_s^{*\cdot} &= \lambda_s \xi_s^* + \mu H_{s1}(\xi^*, \eta^*, M, t) + \mu^2 H_{s2}(\xi^*, \eta^*, N, t) + \ldots \end{aligned} \tag{1.2}$$

obtained from (0.1) by substitution of x_s and y_s from (1.1).

Appearance of the arbitrary constants $M_1, M_2, \ldots$ and $N_1, N_2, \ldots$ in the right sides of (1.2) reflects the fact that x_s^* and y_s^* do not represent some periodic solution already obtained and corresponding to completely defined values of these constants. We know e.g. [2] that the arbitrary constants $M_1, M_2, \ldots$ entering the generating solution $x_{s0}^*(t)$ and $y_{s0}^*(t)$, whose values depend on the character of the roots $\pm i\lambda_s$, as well as the constants appearing in the solutions $x_{s1}^*(t)$, $y_{s1}^*(t), \ldots$ are obtained from the condition of existence of solutions for the next approximation.

However, systems of nonlinear algebraic equations appear even in the case of a system with one degree of freedom, defining the arbitrary constants of the generating solution.

If, on the other hand, we take into account the fact that new periodic or almost periodic oscillations [1] will result in many cases when finite initial perturbations are applied to any periodic solution, then the solution can be obtained by different methods in which the values of arbitrary constants are derived in a much simpler way.

We shall assume that the problem of existence of the finite oscillations is solved in terms of the first approximation in μ, otherwise the transformations analogous to those given below can be extended to the terms containing μ^2 etc., up to μ^α where α is an arbitrarily large number.

Making the substitutions $\zeta_s^* = \xi_s^* + i\eta_s^*$, $\bar{\zeta}_s^* = \xi_s^* - i\eta_s^*$ we pass to the complex variables, obtaining

$$\begin{aligned} \zeta_s^{*\cdot} &= i\lambda_s \zeta_s^* + \mu Z_{s1}^*(\zeta^*, \bar{\zeta}^*, M, t) + \mu^2 Z_{s2}^*(\zeta^*, \bar{\zeta}^*, N, t) + \ldots \\ \bar{\zeta}_s^{*\cdot} &= -i\lambda_s \bar{\zeta}_s^* + \mu \bar{Z}_{s1}^*(\zeta^*, \bar{\zeta}^*, M, t) + \mu^2 \bar{Z}_{s2}^*(\zeta^*, \bar{\zeta}^*, N, t) + \ldots \end{aligned} \tag{1.3}$$

Further we transform the system (1.3) setting

$$\zeta_s^* = \zeta_s + \mu u_{s1}(\zeta^*, \bar{\zeta}^*, t), \qquad \bar{\zeta}_s^* = \bar{\zeta}_s + \mu \bar{u}_{s1}(\zeta^*, \bar{\zeta}^*, t) \tag{1.4}$$

We shall choose the functions u_{s1} and $\bar{u}_{s1}$ with coefficients 2π-periodic in t in such a way, that in the new system

$$\zeta_s^{\cdot} = i\lambda_s \zeta_s + \mu Z_{s1}(\zeta, \bar{\zeta}, M, t) + \ldots, \; \bar{\zeta}_s^{\cdot} = -i\lambda_s \bar{\zeta}_s + \mu \bar{Z}_{s1}(\zeta, \bar{\zeta}, M, t) + \ldots \tag{1.5}$$

the expressions Z_{s1} defined by

$$-\left[\frac{\partial u_{s1}}{\partial t}+\sum_{\beta=1}^{n}\left(\frac{\partial u_{s1}}{\partial \zeta_\beta} i\lambda_\beta\zeta_\beta-\frac{\partial u_{s1}}{\partial \bar\zeta_\beta} i\lambda_\beta\bar\zeta_\beta\right)\right]+i\lambda_s u_{s1}+Z_{s1}^*=Z_{s1} \qquad (1.6)$$

are independent of time. The latter can only be attained when definite conditions imposed on the coefficents of Z_{s1}^* are fulfilled. We shall determine the arbitrary constants $M_1, M_2,\ldots$ in such a way, that these conditions are satisfied. Then we shall be able to make a transition from investigation of nonautonomous system to investigation of autonomous system, with first order terms in μ included.

Let us represent the functions Z_{s1}^* in the form

$$Z_{s1}^*=\sum_{k=1}^{m_s} Z_{s1}^{*(k)}(\zeta^*,\bar\zeta^*,M,t),\qquad Z_{s1}^{*(k)}=\sum_{p=0}^{k}\sum_{\alpha=0}^{p} Z_{s1,k-p}^{*(p-\alpha,\alpha)}\zeta_s^{*p-\alpha}\bar\zeta_s^{*\alpha} \qquad (1.7)$$

Here m_s denotes the highest order form appearing in Z_{s1}^*. Forms $Z_{s1,\,k-p}^{*(p-\alpha,\alpha)}$ do not contain ζ_s^* nor $\bar\zeta_s^*$, and are of order $\delta=k-p$.

We shall assume that the expressions Z_{s1} and u_{s1} have the form analogous to (1.7). Let us substitute the expressions for Z_{s1}^*, Z_{s1} and u_{s1} into (1.6) and compare the coefficients of like powers of $\zeta_s^{p-\alpha}\bar\zeta_s^{\alpha}$. This yields

$$-\frac{\partial u_{s1,k-p}^{(p-\alpha,\alpha)}}{\partial t}-\sum_{\beta=1}^{n}{}'\left(\frac{\partial u_{s1,k-p}^{(p-\alpha,\alpha)}}{\partial\zeta_\beta} i\lambda_\beta\zeta_\beta-\frac{\partial u_{s1,k-p}^{(p-\alpha,\alpha)}}{\partial\bar\zeta_\beta} i\lambda_\beta\bar\zeta_\beta\right)-$$
$$-i\lambda_s(p-2\alpha-1)u_{s1,k-p}^{(p-\alpha,\alpha)}+Z_{s1,k-p}^{*(p-\alpha,\alpha)}=Z_{s1,k-p}^{(p-\alpha,\alpha)} \qquad (1.8)$$

Here and in the following, a prime denotes the fact that terms with the index s are not included in the sum.

Let the forms $\qquad Z_{s1,k-p}^{*(p-\alpha,\alpha)},\quad Z_{s1,k-p}^{(p-\alpha,\alpha)},\quad u_{s1,k-p}^{(p-\alpha,\alpha)}$

of order δ be represented as

$$Z_{s1,k-p}^{*(p-\alpha,\alpha)}=\sum{}' A_{s1}^{(**)}(M,t)\zeta_1^{k_1}\ldots\zeta_n^{k_n}\bar\zeta_1^{q_1}\ldots\bar\zeta_n^{q_n} \qquad (1.9)$$
$$Z_{s1,k-p}^{(p-\alpha,\alpha)}=\sum{}' B_{s1}^{(**)}(M,t)\zeta_1^{k_1}\ldots\zeta_n^{k_n}\bar\zeta_1^{q_1}\ldots\bar\zeta_n^{q_n}$$
$$u_{s1,k-p}^{(p-\alpha,\alpha)}=\sum{}' u_{s1}^{(**)}(t)\zeta_1^{k_1}\ldots\zeta_n^{k_n}\bar\zeta_1^{q_1}\ldots\bar\zeta_n^{q_n}\quad(k_1+\ldots+k_n+q_1+\ldots+q_n=\delta,\ \delta=0,1,\ldots,m)$$

Here and in the following, the asterisk replaces the superscript $(k_1,\ldots,k_n,q_1,\ldots,q_n)$ and m denotes the highest order form appearing in the polynomials X_{s1}, Y_{s1}.

Inserting (1.9) into (1.8) and comparing the coefficients of like powers of $\zeta_1^{k_1}\ldots$ $\ldots\zeta_n^{k_n}\bar\zeta_1^{q_1}\ldots\bar\zeta_n^{q_n}$, we obtain

$$-du_{s1}^{(**)}/dt-in_\nu u_{s1}^{(**)}+A_{s1}^{(**)}=B_{s1}^{(**)} \qquad (1.10)$$

where

$$n_\nu=\sum_{\beta=1}^{n}{}'(k_\beta-q_\beta)\lambda_\beta+\lambda_s(p-2\alpha-1)$$

are various numbers. When resonance occurs, n_ν may assume integral or zero values.

If n_ν is not an integer or zero, the corresponding coefficients $B_{s1}^{(**)}$ can be chosen equal to zero. In this case Eqs. (1.10) will yield $u_{s1}^{(**)}$ in the form of 2π-periodic functions.

If n_ν is an integer, we can choose the coefficients $B_{s1}^{(**)}$ equal to zero, only when the equations

$$-du_{s1}^{(**)}/dt - in_\nu u_{s1}^{(**)} + A_{s1}^{(**)}(M, t) = 0$$

have 2π-periodic solutions for $u_{s1}^{(**)}$. The sufficient condition for this to occur is, that the Fourier expansions

$$A_{s1}^{(**)}(M, t) = \sum_{n=-\infty}^{\infty} A_{s1,n}^{(**)}(M)\, e^{int}$$

contain no coefficients $A_{s1,n}^{(**)}$ for $n = n_\nu$.

In the case of resonance, n_ν are equal to zero not only when $k_\beta = q_\beta$ and $p = 2\alpha + 1$ (self-resonance), but also may be equal to zero for certain $k_\beta \neq q_\beta$ and $p \neq 2\alpha + 1$. The latter will depend on the relations between $\pm i\lambda_s$.

Let us impose the following condition on the coefficients $A_{s1}^{(**)}$ for $n_\nu = 0$ when $k_\beta \neq q_\beta$, $p \neq 2\alpha + 1$:

$$\int_0^{2\pi} A_{s1}^{(**)}(M, t)\, dt = 0$$

From this we have

$$A_{s1,n}^{(**)} = 0 \quad (n = n_\nu), \qquad \int_0^{2\pi} A_{s1}^{(**)} dt = 0 \quad (n_\nu = 0,\ k_\beta \neq q_\beta,\ p \neq 2\alpha + 1) \tag{1.11}$$

and the latter can be used to determine the constants $M_1, M_2, \ldots$ of the generating solution.

We should note that the number of Eqs. (1.11) will, in general, exceed the number of constants $M_1, M_2, \ldots$ However, when some solutions exist in the bounded region, we can pass from one solution to the next by assigning finite values to ξ_{s0}^* and η_{s0}^*. This implies that those constants $M_1, M_2, \ldots$ which satisfy some of Eqs. (1.11), will obviously satisfy all of them.

We shall assume that the values of $M_1, M_2, \ldots$ found, satisfy all Eqs. (1.11) and represent in addition a simple solution of the equations $P_i(M) = 0$ obtained from the conditions of existence of a periodic solution x_{s1}^*, y_{s1}^* for the first approximation.

Let us define the coefficients $B_{s1}^{(**)}$ (in the case of self-resonance) by the equations

$$B_{s1}^{(**)} = \frac{1}{2\pi} \int_0^{2\pi} A_{s1}^{(**)} dt$$

The expressions $Z_{s1}(\zeta, \bar{\zeta}, t)$ appearing in (1.5) will now be time-independent.

First equation of (1.5) can be written as

$$\zeta_s^{\cdot} = i\lambda_s \zeta_s + \mu \zeta_s \sum B_{s1}^{(**)} (\zeta_1 \bar{\zeta}_1)^{k_1} \ldots (\zeta_n \bar{\zeta}_n)^{k_n} + \mu^2 (\ldots) + \ldots \tag{1.12}$$

Returning to the real variables and making the substitution

$$\zeta_s = \xi_s + i\eta_s,\ \bar{\zeta}_s = \xi_s - i\eta_s,\ B_{s1}^{(k_1, \ldots, k_n, q_1, \ldots, q_n)} = a_s^{(k_1, \ldots, k_n)} + ib_s^{(k_1, \ldots, k_n)}$$

we obtain

$$\begin{aligned}
\xi_s^{\cdot} &= -\lambda_s \eta_s + \mu \Big[\xi_s \sum a_s^{(k_1, \ldots, k_n)} (\xi_1^2 + \eta_1^2)^{k_1} \ldots (\xi_n^2 + \eta_n^2)^{k_n} - \\
&\quad - \eta_s \sum b_s^{(k_1, \ldots, k_n)} (\xi_1^2 + \eta_1^2)^{k_1} \ldots (\xi_n^2 + \eta_n^2)^{k_n}\Big] + \mu^2 (\ldots) + \ldots \\
\eta_s^{\cdot} &= \lambda_s \xi_s + \mu \Big[\xi_s \sum b_s^{(k_1, \ldots, k_n)} (\xi_1^2 + \eta_1^2)^{k_1} \ldots (\xi_n^2 + \eta_n^2)^{k_n} + \\
&\quad + \eta_s \sum a_s^{(k_1, \ldots, k_n)} (\xi_1^2 + \eta_1^2)^{k_1} \ldots (\xi_n^2 + \eta_n^2)^{k_n}\Big] + \mu^2 (\ldots) + \ldots
\end{aligned} \tag{1.13}$$

Setting $\xi_s = r_s \cos\theta_s$, $\eta_s = r_s \sin\theta_s$ in (1.13), we have

$$r_s^{\cdot}=\mu r_s \sum a_s^{(k_1,\ldots,k_n)} r_1^{2k_1}\ldots r_n^{2k_n}+\mu^2(\ldots)+\ldots$$
$$\theta_s^{\cdot}=\lambda_s+\mu\sum b_s^{(k_1,\ldots,k_n)} r_1^{2k_1}\ldots r_n^{2k_n}+\mu^2(\ldots)+\ldots \qquad (1.14)$$

Steady-state oscillations which include the periodic oscillations for the present system correct to first order terms in μ, are obtained from the solutions of the following algebraic equations

$$r_s \sum a_s^{(k_1,\ldots,k_n)} r_1^{2k_1}\ldots r_n^{2k_n}=0 \qquad (1.15)$$

When the highest order form in X_{s1} and Y_{s1} is not greater than cubic, we can easily see that the system (1.15) reduces to a system of linear algebraic equations. If, on the other hand, X_{s1} and Y_{s1} contain forms of order higher than the third, the system (1.15) is still much simpler than the nonlinear system $P_i(M)=0$.

Determining the positive solutions r_{s0}^2 of the system (1.15), we can obtain new values for the constants $M_1, M_2, \ldots$.

Indeed, expressing the variables ξ_s, η_s by

$$\xi_s = r_{s0}\cos\theta_s, \quad \eta_s = r_{s0}\sin\theta_s$$

and returning to the variables ξ_s^*, η_s^*, we have

$$\xi_s^* = r_{s0}\cos\theta_s+\mu(\ldots)+\ldots, \quad \eta_s^*=r_{s0}\sin\theta_s+\mu(\ldots)+\ldots$$

Solutions (1.1) correct to the first order terms in μ take the form

$$x_s=x_{s0}^*+r_{s0}\cos\theta_s+\mu(\ldots)+\ldots, \quad y_s=y_{s0}^*+r_{s0}\sin\theta_s+\mu(\ldots)+\ldots \qquad (1.16)$$

In the terms not containing μ the values θ_s are equal to $\lambda_s t$.

The family of periodic solutions in the generating system includes the sum of terms $M_s \sin\lambda_s t$ and $M_s\cos\lambda_s t$, therefore we can amalgamate these terms with the corresponding $r_{s0}\sin\lambda_s t$ and $r_{s0}\cos\lambda_s t$ obtaining new values of the arbitrary constants equal to the sum of M_s and r_{s0}.

In general, the solutions (1.16) can be periodic or almost-periodic, depending on the character of the roots $\pm i\lambda_s$.

Arbitrary constants entering the solutions x_{s1}^*, y_{s1}^* can be found by considering the next approximation.

We also note that the stability of the solutions obtained can be inspected very simply when using the method given in [1].

2. Examples. 1. Our first example will concern a second kind resonance in a regenerative receiver. The problem has been studied in detail in paper [3].

Equation of oscillations has the form

$$x^{\cdot\cdot}+x=\mu(1-x^2)x^{\cdot}+\lambda\sin 2t \qquad (2.1)$$

Setting $x^{\cdot}=-y$, we can write (2.1) as

$$x^{\cdot}=-y, \qquad y^{\cdot}=x+\mu(1-x^2)y-\lambda\sin 2t \qquad (2.2)$$

The generating solution has the form

$$x_0=M\cos t+N\sin t-{}^1/_3\,\lambda\sin 2t, \quad y_0=M\sin t-N\cos t+{}^2/_3\,\lambda\cos 2t$$

Equations defining the arbitrary constants M and N obtained from the conditions of existence of periodic solutions for the first approximation, are

$$M\,[{}^1/_{18}\,\lambda^2-1+{}^1/_4\,(M^2+N^2)]=0, \qquad N\,[{}^1/_{18}\,\lambda^2-1+{}^1/_4\,(M^2+N^2)]=0 \quad (2.3)$$

We shall employ the method given previously, to obtain the values of M and N without solving (2.3).

Writing (2.2) in the form of (1.2), we obtain

$$\xi^{*\cdot} = -\eta^*, \quad \eta^{*\cdot} = \xi^* - \mu\,[2x_0y_0\xi^* + \eta^*\,(x_0^2 - 1) + y_0\xi^{*2} + 2x_0\xi^*\eta^* + \xi^{*2}\,\eta^*]$$

Passing further to the complex variables and writing equations of the type (1.10), we can determine the values of M and N. Comparing the coefficients of $\bar{\zeta}$, we have

$$-u^{\cdot(0.1)} + 2iu^{(0.1)} + A^{(0.1)} = 0 \qquad (2.4)$$

where $A^{(0.1)} = {}^1/_2\,(a - ib), \quad a = x_0^2 - 1, \quad b = 2x_0y_0$.

This will have a periodic solution for $u^{(0.1)}$, if the coefficient $A_n^{(0.1)}$ in the Fourier expansion

$$\overset{\bullet}{A}{}^{(0.1)} = \sum_n A_n^{(0.1)}\,(M, N)\,e^{int}$$

is equal to zero for $n = 2$.

Condition $A_2^{(0.1)} = 0$ will hold, if

$${}^1/_4\,(N^2 - M^2) = 0, \quad NM = 0$$

This yields $M = N = 0$, which is a simple solution of (2.3).

Performing further the transformations given in Sect. 1 we obtain in the place of (1.15),

$$r\,(18 - \lambda^2 - {}^9/_2 r^2) = 0, \qquad r_{10} = 0, \; r_{20} = \sqrt{4 - {}^2/_9\lambda^2}$$

In the present case solutions (1.16) have the form

$$x = -{}^1/_3\,\lambda \sin 2t + r_{20}\cos t + \mu\,(\ldots) + \ldots, \quad y = {}^2/_3\,\lambda\cos 2t + r_{20}\sin t + \mu\,(\ldots) + \ldots$$

M and N acquire new values given by

$$M = r_{20} = \sqrt{4 - {}^2/_9\lambda^2}, \; N = 0$$

and the generating solution has the form

$$x_0 = -{}^1/_3\,\lambda\sin 2t + \sqrt{4 - {}^2/_9\lambda^2}\cos t, \quad y_0 = {}^2/_3\lambda\cos 2t + \sqrt{4 - {}^2/_9\lambda^2}\sin t$$

Stability of the solutions obtained can be examined very simply, using the method given in [1]. We require the value of the coefficient

$$p\,(r) = d\,[r\,(18 - \lambda^2 - {}^9/_2\,r^2)]\,/\,dr = 18 - \lambda^2 - {}^{27}/_2\,r^2$$

The necessary condition for the solution $r_{10} = 0$ to be stable is

$$p\,(0) = 18 - \lambda^2 < 0, \; \lambda^2 > 18$$

and for the solution $r_{20} = \sqrt{4 - {}^2/_9\,\lambda^2}$ is

$$p\,(r_{20}) = 2\,(\lambda^2 - 18) < 0, \; \lambda^2 < 18$$

We note, that in this simple example, all possible values of M and N (and not only the ones obtained above) are easily found from the system (2.3). In addition to $M = N = 0$, any M and N satisfying the equation

$$M^2 + N^2 = 4 - {}^2/_9\,\lambda^2$$

will be the solutions of (2.3).

Serious difficulties arise, however, when solution of systems analogous to (2.3) is attempted for a system more complicated than (2.1) even with one degree of freedom. Difficulties may also be encountered when variational equations are used to examine the stability of the solutions obtained. For example, in the present case variational equations yield the condition $\lambda^2 > 18$ for the stability of the solution $M = N = 0$. However, when variational equations are used to investigate the stability of any of the remaining solutions, no results emerge, since the corresponding characteristic equation has a single root equal to unity. The difficulties increase even more in the case of systems with more than one degree of freedom.

The next example will illustrate the point.

2. In the second example we shall investigate the oscillations of an electronic generator [4] acted upon by a complex external force (defined by a set of sine and cosine harmonics resonating with the natural frequencies of the generator).

Performing the transformations given in [4], we obtain the following system of equations:

$$x_1^{\cdot} = -y_1, \quad y_1 = x_1 + \mu\varphi(x_1, x_2, y_1, y_2) + P_2 \sin 2t + \mu Q_2 \cos t$$

$$x_2^{\cdot} = -2y_2 + \mu f(x_1, x_2, y_1, y_2) + P_1 \sin t + \mu Q_1 \cos 2t, \quad y_2^{\cdot} = 2x_2 \tag{2.5}$$

In this system $\lambda_1 = 1$, $\lambda_2 = 2$ we have both internal and external resonance. Functions f and φ can be written in the form

$$f(x_1, x_2, y_1, y_2) = a_0 y_1 - a_1 x_1^2 y_1 - a_2 x_1 y_1 y_2 - a_3 y_1 y_2^2 - a_4 x_2 + a_5 x_1^2 x_2 + a_6 x_1 x_2 y_2 + a_7 x_2 y_2^2 \tag{2.6}$$

$$\varphi(x_1, x_2, y_1, y_2) = b_0 y_1 - b_1 x_1^2 y_1 - b_2 x_1 y_1 y_2 - b_3 y_1 y_2^2 - b_4 x_2 + b_5 x_1^2 x_2 + b_6 x_1 x_2 y_2 + b_7 x_2 y_2^2$$

Coefficients $a_0, \ldots, a_7, b_0, \ldots, b_7$ were obtained in terms of the parameters of the system (inductance, capacitance, resistance, etc.) in [4], and have the form

$$a_0 = \left(\frac{1}{n_1^2} + \frac{r}{n_2^2}\right)\frac{k_2\omega_2^2}{q}, \quad a_1 = \frac{k_2\omega_2^2}{\omega_1^2\gamma}, \quad a_2 = \frac{2k_1k_2\omega_2}{\omega_1\gamma}, \quad a_3 = \frac{k_1^2k_2}{\gamma}$$

$$a_4 = \left(\frac{k_1k_2}{n_1^2} + \frac{r}{n_2^2}\right)\frac{\omega_2^2}{q}, \quad a_5 = k_1a_1, \quad a_6 = k_1a_2, \quad a_7 = k_1a_3$$

$$b_0 = \left(\frac{1}{n_1^2} + \frac{k_1k_2r}{n_2^2}\right)\frac{\omega_1^2}{q}, \quad b_1 = \frac{1}{\gamma}, \quad b_2 = \frac{2k_1\omega_1}{\omega_2\gamma}, \quad b_3 = \frac{\omega_1^2k_1^2}{\omega_2^2\gamma}$$

$$b_4 = \left(\frac{1}{n_1^3} + \frac{r}{n_2^2}\right)\frac{k_1\omega_1^2}{q}, \quad b_5 = k_1b_1, \quad b_6 = k_1b_2, \quad b_7 = k_1b_3$$

$$\omega_1 = \lambda_1, \quad \omega_2 = \lambda_2, \quad \gamma = n_1^2q^3, \quad q = 1 - k_1k_2$$

External force acting on the system can be described by the coefficients P_1, Q_1, P_2 and Q_2 as follows:

$$P_1 = \frac{\omega_2^2A_1}{k_1n_1^2}(k_1k_2 - 1), \qquad P_2 = \frac{\omega_1^2A_2}{n_1^2}(1 - k_1k_2)$$

$$Q_1 = -\frac{8r}{3n_2^2k_1}P_2, \qquad Q_2 = \frac{k_1r}{3n_2^2}P_1$$

The values of A_1 and A_2 are defined in terms of amplitudes of the harmonics of the external force, while r, $k_1, \ldots$ – in terms of parameters of the oscillating system.

We shall seek the periodic solution of the system (2.5) in the form of Poincaré series

$$x_1^* = x_{10} + \mu x_{11} + \ldots, \quad y_1^* = y_{10} + \mu y_{11} + \ldots$$

$$x_2^* = x_{20} + \mu x_{21} + \ldots, \quad y_2^* = y_{20} + \mu y_{21} + \ldots \tag{2.7}$$

where $x_{10}, x_{20}, y_{10}, y_{20}$ are solutions of the generating system

$$x_{10}^{\cdot} = -y_{10}, \quad x_{20}^{\cdot} = -2y_{20} + P_1 \sin t$$

$$y_{10}^{\cdot} = x_{10} + P_2 \sin 2t, \quad y_{20}^{\cdot} = 2x_{20}$$

equal to

$$x_{10} = {}^1/_3\, P_2 \sin 2t + M_1 \sin t + N_1 \cos t, \quad x_{20} = {}^1/_3\, P_1 \cos t + M_2 \cos 2t - N_2 \sin 2t$$

$$y_{10} = -{}^2/_3\, P_2 \cos 2t - M_1 \cos t + N_1 \sin t, \quad y_{20} = {}^2/_3\, P_1 \sin t + M_2 \sin 2t + N_2 \cos 2t$$

Following the method given in [2], we obtain the values of M_1, N_1 and M_2, N_2 from the conditions of existence of periodic solutions for the first approximation equations

$$x_{11}' = -y_{11}, \quad x_{21}' = -2y_{21} + f(x_{10}, x_{20}, y_{10}, y_{20}) + Q_1 \cos 2t$$
$$y_{11}' = x_{11} + \varphi(x_{10}, x_{20}, y_{10}, y_{20}) + Q_2 \cos t, \quad y_{21}' = 2x_{21}$$

These lead to the following nonlinear system of algebraic equations defining M_1, N_1 and M_2, N_2:

$$\begin{gathered}
27 M_1 [b_1 (M_1^2 + N_1^2) + 2b_3 (M_2^2 + N_2^2)] + 9P_1 (3b_5 - 2b_2) N_1^2 + \\
+ 9P_1 (2b_2 + b_5) M_1^2 + 18b_7P_1 (M_2^2 + N_2^2) + 36 P_2 (b_2 - b_5) N_1N_2 + \\
+ 18b_2P_2M_1M_2 + 6 [-18 b_0 + b_1P_2^2 + P_1^2 (2b_3 + b_6)] M_1 + 6b_6 P_1P_2M_2 + \\
+ 2 (-18 b_4P_1 + b_5P_1P_2^2 + 2b_7P_1^3 + 54 Q_2) = 0 \\
-9N_1 [b_1 (M_1^2 + N_1^2) + 2b_3 (M_2^2 + N_2^2)] + 6P_1M_1N_1 (b_5 - 2b_2) - \\
- 6b_2P_2N_1M_2 + 12P_2M_1N_2 (b_2 - b_5) + 12b_7P_1M_2N_2 + 2N_1 [18b_0 - \\
-b_1P_2^2 + P_1^2 (b_6 - 6b_3)] + 4P_1P_2N_2 (4b_3 - b_6) = 0 \qquad (2.8) \\
-9N_2 [2a_5 (M_1^2 + N_1^2) + a_7 (M_2^2 + N_2^2)] + 6P_1N_1M_2 (a_6 - 4a_2) + \\
+ 6P_2M_2N_2(2a_3 - a_6) - 12a_6P_1M_1N_2 + 4P_1P_2N_1 (a_5 - a_2) + \\
+ N_2 [36a_4 + P_2^2 (2a_3 - 3a_5) - 8a_7P_1^2] = 0 \\
27M_2 [2a_5 (M_1^2 + N_1^2) + a_7(M_2^2 + N_2^2)] + 36a_1P_2 (M_1^2 + N_1^2) + \\
+ 18P_1N_1N_2 (a_6 - 4a_2) + 9P_2M_2^2 (a_6 + 2a_3) + 9P_2N_2^2 (6a_3 - a_6) + \\
+ 36a_6P_1M_1M_2 + 24a_2P_1P_2M_1 + 3M_2 [P_2^2 (2a_3 + a_5) - 36 a_4 + \\
+ 8a_7P_1^2] + 2 (-36a_0P_2 + a_1P_2^3 + 8a_2P_1^2P_2 + 54Q_1) = 0
\end{gathered}$$

We shall seek the solution, following the method given above. Transforming the system (2.5) into the form of (1.2), we obtain

$$\xi_1^{*\cdot} = -\eta_1^*, \quad \xi_2^{*\cdot} = -2\eta_2^* + \mu\, \Xi_{21} (\xi_1^*, \xi_2^*, \eta_1^*, \eta_2^*, M_1, M_2, N_1, N_2, t) + \ldots$$
$$\eta_1^{*\cdot} = \xi_1^* + \mu H_{11} (\xi_1^*, \eta_1^*, \xi_2^*, \eta_2^*, M_1, M_2, N_1, N_2, t) + \ldots, \quad \eta_2^{*\cdot} = 2\xi_2^* \qquad (2.9)$$

$$\begin{gathered}
\Xi_{21} = \xi_1^* (2a_5x_{10}x_{20} + a_6x_{20}y_{20} - 2a_1x_{10}y_{10} - a_2y_{10}y_{20}) + \\
+ \xi_2^* (a_5x_{10}^2 + a_6x_{10}y_{20} + a_7y_{20}^2 - a_4) + \eta_1^* (a_0 - a_1x_{10}^2 - \\
-a_2x_{10}y_{20} - a_3y_{20}^2) + \eta_2^* (a_6x_{10}x_{20} + 2a_7x_{20}y_{20} - a_2x_{10}y_{10} - \\
-2a_3y_{10}y_{20}) + \xi_1^{*2} (a_5x_{20} - a_1y_{10}) + \eta_2^{*2} (a_7x_{20} - a_3y_{10}) - \\
-\xi_1^*\eta_1^* (2a_1x_{10} + a_2y_{20}) + \xi_1^*\eta_2^* (a_6x_{20} - a_2y_{10}) - \\
-\eta_1^*\eta_2^* (a_2x_{10} + 2a_3y_{20}) + \xi_1^*\xi_2^* (2a_5x_{10} + a_6y_{20}) + \\
+\xi_2^*\eta_2^* (a_6x_{10} + 2a_7y_{20}) - a_1\xi_1^{*2}\eta_1^* - a_2\xi_1^*\eta_1^*\eta_2^* - a_3\eta_1^*\eta_2^{*2} + \\
+ a_5\xi_1^{*2}\xi_2^* + a_6 \xi_1^*\xi_2^*\eta_2^* + a_7\eta_2^{*2}\xi_2^*
\end{gathered}$$

Function H_{11} can be obtained from Ξ_{21} by replacing the coefficients $a_0, \ldots, a_7$ by $b_0, \ldots, b_7$, respectively.

Transforming the system (2.9) further according to Sect. 1, we arrive at equations of the form (1.10). In the present case all n_v are either zero or integers. One of these equations (for $k = 2$ at $\zeta_1\bar{\zeta}_1$) is

$$-u_{11.0}^{\cdot(1.1)} + iu_{11.0}^{(1.1)} + Z_{11.0}^{*(1.1)} = 0, \quad Z_{11.0}^{*(1.1)}\Big|_{n=1} = -{}^1/_2 ib_1N_1 \sin t +$$
$$+ {}^1/_8 i \cos t\, (3b_1M_1 + b_5P_1)$$

Condition of existence of a periodic solution for $u_{11.0}^{(1.1)}$ yields

$$N_1 = 0, \qquad {}^1/_8 (3b_1M_1 + b_5P_1) = 0$$

Taking now into account the fact that $b_1 = 1 / n_1^2q^2$ and $b_5 = k_1 / n_1^2q^2$, we obtain $M_1 = -{}^1/_3\, k_1P_1$.

For $k = 2$ at $\zeta_1\zeta_2$ we obtain

$$-u_{11}^{\cdot\cdot(0.1.0.0)} - 2iu_{11}^{\cdot(0.1.0.0)} + A_{11}^{*(0.1.0.0)} = 0$$

$$A_{11}^{*(0.1.0.0)}\Big|_{n=2} = -{}^1/_4\, b_6N_2 \sin 2t + ({}^1/_6\, b_2P_2 + {}^1/_4\, b_6M_2)\cos 2t +$$
$$+ ({}^1/_{12}\, b_2P_2 + {}^1/_2\, b_2M_2 + {}^1/_8\, b_5P_2 + {}^1/_4\, b_6M_2)\, t \sin 2t +$$
$$+ ({}^1/_2 b_2N_2 + {}^1/_4\, b_6N_2)\, t \cos 2t$$

Condition of existence of a periodic solution for $u_{11}^{(0.1.0.0)}$ yields

$$N_2 = 0, \qquad M_2 = -{}^2/_3\, P_2 / k_1$$

since $b_5 = k_1 / n_1^2q^3$ and $b_2 = {}^1/_4\, k_1^2 / n_1^2q^3$.

The obtained values M_1, M_2 and N_1, N_2 satisfy Eqs. (2.8).

After further transformations we arrive at a system of the form (1.12) which in the present case has the form

$$\zeta_1^{\cdot} = i\zeta_1 + \mu\zeta_1(a_{10} + a_{11}\zeta_1\overline{\zeta}_1 + a_{12}\zeta_2\overline{\zeta}_2) + \mu^2(\ldots) + \ldots$$
$$\zeta_2^{\cdot} = 2i\zeta_2 + \mu\zeta_2(a_{20} + a_{21}\zeta_1\overline{\zeta}_1 + a_{22}\zeta_2\overline{\zeta}_2) + \mu^2(\ldots) + \ldots \qquad (2.10)$$

$$a_{10} = {}^1/_2\, b_0,\ a_{11} = -{}^1/_8 b_1,\ a_{12} = -{}^1/_4\, b_3,\ a_{20} = -{}^1/_2 a_6,\ a_{21} = {}^1/_4 a_5,\ a_{22} = {}^1/_8\, a_7$$

while the system (1.14) becomes

$$r_1^{\cdot} = \mu r_1(a_{10} + a_{11}r_1^2 + a_{12}r_2^2) + \mu^2(\ldots) + \ldots,\quad \theta_1^{\cdot} = 1 + \mu^2(\ldots) + \ldots$$
$$r_2^{\cdot} = \mu r_2(a_{20} + a_{21}r_1^2 + a_{22}r_2^2) + \mu^2(\ldots) + \ldots,\quad \theta_2^{\cdot} = 2 + \mu^2(\ldots) + \ldots \qquad (2.11)$$

The equations

$$r_1(a_{10} + a_{11}r_1^2 + a_{12}r_2^2) = 0, \qquad r_2(a_{20} + a_{21}r_1^2 + a_{22}r_2^2) = 0$$

are reduced to linear algebraic equations by means of the substitution $r_1^2 = \gamma_1$ and $r_2^2 = \gamma_2$, and their solutions are

(1) $r_1 = 0, r_2 = 0$ (2) $r_2 = 0, r_1 = 2\sqrt{b_0/b_1}$, (3) $r_1 = 0, r_2 = 2\sqrt{a_4/a_7}$

(4) $r_1 = \sqrt{4(2a_4b_3 - a_7b_0)/(4a_5b_3 - a_7b_1)}$, $r_2 = \sqrt{4(2a_5b_0 - a_4b_1)/(4a_5b_3 - a_7b_1)}$

When $r_1 = r_2 = 0$, we arrive at the previously obtained values for the arbitrary constants of the generating solution, and using the second, third and fourth solutions, we obtain new values for M_1, M_2 and N_1, N_2 which are respectively

1) $M_1 = -{}^1/_3 k_1P_1$, $M_2 = -{}^2/_3 P_2/k_1$, $N_1 = 0$, $N_2 = 0$

2) $M_1 = -{}^1/_3 k_1P_1$, $M_2 = -{}^2/_3 P_2/k_1$, $N_1 = 2\sqrt{b_0/b_1}$, $N_2 = 0$

3) $M_1 = -{}^1/_3 k_1P_1$, $M_2 = -{}^2/_3 P_2/k_1 + 2\sqrt{a_4/a_7}$, $N_1 = 0$, $N_2 = 0$

4) $M_1 = -{}^1/_3 k_1P_1$, $M_2 = \sqrt{4(2a_5b_0 - a_4b_1)/(4a_5b_3 - a_7b_1)} - {}^2/_3 P_2/k_1$

$N_1 = \sqrt{4(2a_4b_3 - a_7b_0)/(4a_5b_3 - a_7b_1)}$, $N_2 = 0$

The method developed in [1] can again be used to investigate the stability of the new solutions in a simple manner.

In conclusion we note that the method given in Sect. 1 yields the constants of the generating solution as functions of the parameters of the system and of the external forces. This becomes particularly important in the process of analyzing oscillatory systems.

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ON THE LIBRATION BOUNDARIES OF A SATELLITE IN CIRCULAR ORBIT UNDER THE ACTION OF POTENTIAL PERTURBING FORCES

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The parameters of the final rotation of a satellite with respect to its position of stable equilibrium are chosen as variables convenient for estimating the potential energy of the perturbing forces. It is shown that the perturbing forces and deviations of the satellite satisfy inequalities (3.4) and (3.6). These inequalities constitute the conditions of (λ, A, t_0, T)- stability [1] of the satellite's equilibrium.

1. Let us assume that the center of mass of a satellite moves as a material point along a Keplerian circular orbit and let us introduce the right-handed rectangular coordinate systems $Ox_1x_2x_3$ and $Oy_1y_2y_3$. We direct the axes of the first of these systems along the principal central axes of inertia of the satellite. The second system is the orbital system (y_1 lies along the velocity, y_2 along the normal to the orbital plane, y_3 along the radius vector).

The potential energy of the gravitational and inertial forces acting on the satellite is given by the expression [3]

$$W = a\alpha_{21}^2 + b\alpha_{23}^2 + c\alpha_{31}^2 + d\alpha_{32}^2, \quad \alpha_{ij} = \cos y_i x_j \tag{1.1}$$

$$a = {}^1/_2\omega^2 (A_2 - A_1), \; b = {}^1/_2 \omega^2(A_2 - A_3), \; c = {}^3/_2 \omega^2 (A_1 - A_3), \; d = {}^3/_2 \omega^2 (A_2 - A_3)$$

in the orbital coordinate system. Here ω is the Keplerian orbital angular velocity and A_i are the principal central moments of inertia of the satellite. The coefficients a, b, c, d are related to each other by the self-evident equations

$$d = 3b = c + 3a \tag{1.2}$$

The relative motions of the satellite in the orbital coordinate system have the energy integral H,

$$H = T + W = h, \quad 2T = A_1p_1^2 + A_2p_2^2 + A_3p_3^2 \tag{1.3}$$

Here T is the kinetic energy of the relative motions and p_i are the projections of the relative angular velocity of the satellite onto the axes x_i.

The table of cosines α_{ij} expressed in terms of the Rodrigues-Hamilton parameters λ_0, λ_i ($i = 1, 2, 3$) can be written out in the following form:

	x_1	x_2	x_3
y_1	$\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2$	$2(-\lambda_0\lambda_3 + \lambda_1\lambda_2)$	$2(\lambda_0\lambda_2 + \lambda_1\lambda_3)$
y_2	$2(\lambda_0\lambda_3 + \lambda_1\lambda_2)$	$\lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2$	$2(-\lambda_0\lambda_1 + \lambda_2\lambda_3)$
y_3	$2(-\lambda_0\lambda_2 + \lambda_1\lambda_3)$	$2(\lambda_0\lambda_1 + \lambda_2\lambda_3)$	$\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2$

$$\lambda_0 = \cos {}^1/_2\, \chi, \qquad \lambda_i = \gamma_i \sin {}^1/_2\, \chi \qquad (i = 1, 2, 3), \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$